CERTAIN ELASTOPLASTIC PROBLEMS FOR A PLANE WEAKENED BY A DOUBLY PERIODIC SYSTEM OF CIRCULAR HOLES

V. M. Mirsalimov

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Elastoplastic problems for an infinite perforated plane possessing a square grid of circular holes are examined. It is assumed that the level of stresses and the grid spacing are such that the circular holes envelop the corresponding plastic zone, though neighboring plastic regions are not connected. The elastoplastic problem for a triangular grid has been previously examined [1] under the assumption that the matter in the elastic and plastic regions is homogeneous.

Suppose we have a doubly periodic orthogonal grid with circular holes with radius $R (R < 1)$ and centers at the points

$$P_{mn} = m\omega_1 + n\omega_2 \quad (m, n = 0, \pm 1, \pm 2, \ldots)$$
$$\omega_1 = 2, \ \omega_2 = 2i$$

We denote the circumference of a hole with center at P_{mn} by L_{mn}, the corresponding elastoplastic boundary by Γ_{mn}, and the exterior of the circumferences Γ_{mn} by D_z.

Boundary conditions on the circumference of the holes L_{mn} have the form

$$\sigma_r = -p, \ \tau_{r\theta} = 0 \tag{1}$$

We will assume that the stress field in the plastic zone has the form

$$\sigma_r = \frac{A}{r^2} + B(1 + 2\ln r) + 2C,$$
$$\sigma_\theta = -\frac{A}{r^2} + B(3 + 2\ln r) + 2C, \quad \tau_{r\theta} = 0 \tag{2}$$

where A, B, and C are given constants.

The axisymmetrical stress field (2), which satisfies balance equations, is characterized by the fact that the constants corresponding to it allow particular plasticity conditions to be satisfied (cf. below) which take into account plastic inhomogeneity, i.e., the dependence of the yield limit on coordinate r and on the principle stresses σ_θ and σ_r. At the same time, a previous method [1] for such a stress field, which combined a method of solving a doubly periodic elastic problem [2] with a method proposed in [3] for solving problems arising in elasticity and plasticity theory, given an unknown boundary with the identical holes, admits of an effective solution of the elastoplastic problem.

All these stresses are continuous on the unknown circumference Γ_{mn} which divides the elastic and plastic regions. Using Eq. (2) and the Kolosov–Muskhelishvili relation [4] we obtain conditions on the circumference Γ_{00}

$$\operatorname{Re}\Phi(z) = \frac{1}{2}B\ln z\bar{z} + (B + C), \quad \bar{z}\Phi'(z) + \Psi(z) = B\frac{\bar{z}}{z} - \frac{A}{z^2} \tag{3}$$

We pass to a parametric plane ξ by means of the transformation $z = \omega(\xi)$. The analytic function $z = \omega(\xi)$ carries out a conformal mapping of the region D_z onto D_ξ in the plane ξ, which is the exterior of the circles γ_{mn} of radius λ with centers at the points P_{mn}.

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To determine the three analytic functions ($\varphi(\zeta) = \Phi[\omega(\zeta)]$, $\psi(\zeta) = \Psi[\omega(\zeta)]$, and $\omega(\zeta)$) we obtain a nonlinear boundary-value problem on γ_{00}

$$\operatorname{Re}\varphi(\zeta) = B + C + \frac{1}{2} B \ln \omega(\zeta)\overline{\omega(\zeta)} \tag{4}$$

$$\frac{\overline{\omega(\zeta)}}{\overline{\omega'(\zeta)}} \varphi'(\zeta) + \psi(\zeta) = B\frac{\overline{\omega(\zeta)}}{\omega(\zeta)} - \frac{A}{[\omega(\zeta)]^2} \tag{5}$$

We find by solving the Dirichlet boundary value problem (4) that the region D_ζ is characterized by

$$\varphi(\zeta) = B + C + \frac{1}{2} B \ln \omega(\zeta) - \frac{1}{2} B \ln \frac{\zeta}{\lambda} \tag{6}$$

The boundary condition (5) can be transformed by taking into account Eq. (6), to the form

$$\omega'(\zeta)\,\omega^2(\zeta)\,\psi(\zeta) = B\frac{\overline{\omega(\zeta)}}{\zeta}\,\omega^2(\zeta) - A\omega'(\zeta) \tag{7}$$

The required functions will be found in the form of series [1, 2]

$$\varphi(\zeta) = \alpha_0 + \sum_{k=0}^{\infty} \alpha_{2k+2} \frac{\lambda^{2k+2}\gamma^{(2k)}(\zeta)}{(2k+1)!} \tag{8}$$

$$\psi(\zeta) = \beta_0 + \sum_{k=0}^{\infty} \beta_{2k+2} \frac{\lambda^{2k+2}\gamma^{(2k)}(\zeta)}{(2k+1)!} - \sum_{k=0}^{\infty} \alpha_{2k+2} \frac{\lambda^{2k+2}Q^{(2k+1)}(\zeta)}{(2k+1)!} \tag{9}$$

$$\omega(\zeta) = \zeta + \sum_{k=0}^{\infty} A_{2k+2} \frac{\lambda^{2k+2}\gamma^{(2k-1)}(\zeta)}{(2k+1)!} \tag{10}$$

where $\gamma(\zeta)$ is the Weierstrass elliptic function and $Q(\zeta)$ a special meromorphic function, defined by

$$\gamma(\zeta) = \frac{1}{\zeta^2} + \sum_{m,n}{}' \left[\frac{1}{(\zeta - P_{mn})^2} - \frac{1}{P_{mn}^2}\right]$$

$$Q(\zeta) = \sum_{m,n}{}' \left[\frac{\bar{P}_{mn}}{(\zeta - P_{mn})^2} - 2\zeta\frac{\bar{P}_{mn}}{P_{mn}^3} - \frac{\bar{P}_{mn}}{P_{mn}^2}\right]$$

$$(m, n = 0, \pm 1, \pm 2, \ldots)$$

Let us present the dependences which the coefficients of the representations (8)-(10) must satisfy. If the principal vector describing the forces acting on an arc joining two congruent points in D_ζ is set equal to zero, we have

$$\alpha_0 = \frac{\pi}{8}\lambda^2\beta_2, \quad \beta_0 = 0 \tag{11}$$

Symmetry conditions on as square grid lead to the relations

$$\beta_{4k} = A_{4k+2} = 0 \quad \text{when } k = 0, 1, \ldots \tag{12}$$

To compile the function for the remaining coefficients of Eqs. (8)-(10) for the functions $\varphi(\zeta)$, $\psi(\zeta)$, and $\omega(\zeta)$, we decompose these functions in a Laurent series in the neighborhood of the point $\zeta = 0$, that is

$$\varphi(\zeta) = \alpha_0 + \sum_{k=0}^{\infty} \alpha_{4k}\left(\frac{\lambda}{\zeta}\right)^{4k} + \sum_{k=1}^{\infty} \alpha_{4k}\lambda^{4k} \sum_{j=0}^{\infty} r_{2j,\,2k-1}\zeta^{4j} \tag{13}$$

$$\psi(\zeta) = \sum_{k=0}^{\infty} \beta_{4k+2}\left(\frac{\lambda}{\zeta}\right)^{4k+2} + \sum_{k=0}^{\infty} \beta_{4k+2}\lambda^{4k+2} \sum_{j=0}^{\infty} r_{2j+1,\,2k}\zeta^{4j+2} - \sum_{k=0}^{\infty} 4k\alpha_{4k}\lambda^{4k} \sum_{j=0}^{\infty} S_{2j+1,\,2k-1}\zeta^{4j+2} \tag{14}$$

$$\omega(\zeta) = \zeta - \sum_{k=1}^{\infty} \frac{A_{4k}\lambda}{(4k-1)}\left(\frac{\lambda}{\zeta}\right)^{4k-1} + \sum_{k=1}^{\infty} A_{4k}\lambda^{4k} \sum_{j=0}^{\infty} \frac{r_{2j,\,2k-1}}{4j+1}\zeta^{4j+1}$$

$$r_{j,k} = \frac{(2j+2k+1)!\,g_{j+k+1}}{(2j)!\,(2k+1)!\,2^{2j+2k+2}}, \quad g_j = \sum_{m,n}{}' \frac{1}{T^{2j}} \tag{15}$$

$$S_{j,k} = \frac{(2j+2k+2)!\,\rho_{j+k+1}}{(2j)!\,(2k+1)!\,2^{2j+2k+2}}, \quad \rho_j = \sum_{m,n}{}' \frac{\bar{T}}{T^{2j+1}}, \quad T = \frac{P_{mn}}{2}$$

Substituting for $\varphi(\zeta)$, $\psi(\zeta)$, and $\omega(\zeta)$ their decompositions in Laurent series in the boundary conditions (4) and (7) on the circumference γ_{00} ($\zeta = \lambda e^{i\theta}$) and comparing coefficients in $e^{i4k\theta}$ ($k = 0, 1, 2, \ldots$), we obtain an infinite system of nonlinear algebraic equations in α_{4k}, β_{4k+2}, and A_{4k} [condition (4) was first differentiated with respect to θ]. The equations of the first approximation are

$$
\begin{aligned}
&Xc_1 + Yc_2 + Zc_3 = B(ac_1 + c_2 b + c_3 c_4) - \frac{Aa}{\lambda^2} \\
&Xc_3 + Yc_1 = B(bc_1 + ac_3) - \frac{AA_4}{\lambda^2} \\
&Xc_2 + Zc_1 = B(c_1 c_4 + ac_2) - AA_4 r_{2,1}\lambda^6, \quad \alpha_4(1 + \lambda^8 r_{2,1})d = -Bd_1 \\
&X = a\beta_2 + A_4\gamma_0 + A_4\beta_6\lambda^8 r_{2,1}, \qquad Y = a\beta_6 + A_4\beta_2 \\
&Z = a\gamma_0 + A_4\gamma_1 + A_4\beta_2\lambda^8 r_{2,1}, \qquad a = 1 + A_4\lambda^4 r_{0,1} \\
&c_1 = a^2 - \frac{2A_4^2\lambda^8 r_{2,1}}{15}, \qquad c_2 = \frac{2}{5} aA_4\lambda^8 r_{2,1} \\
&c_3 = -\frac{2}{3} aA_4, \qquad c_4 = -\frac{1}{3} A_4, \qquad b = \frac{1}{5} A_4\lambda^8 r_{2,1} \\
&d = a^2 + A_4^2\left(\frac{1}{9} + \frac{1}{25}\lambda^{16} r_{2,1}^2\right), \qquad d_1 = aA_4\left(\frac{1}{3} - \frac{1}{5}\lambda^8 r_{2,1}\right) \\
&\gamma_j = \beta_2 r_{2j+1,0}\lambda^{4j+4} + \beta_6 r_{2j+1,2}\lambda^{4j+8} - 4\alpha_4 S_{2j+1,1}\lambda^{4j+6} \quad (j = 0, 1)
\end{aligned}
\tag{16}
$$

We substitute Eqs. (8) and (10) in the boundary condition (4), multiply the resulting expression by $1/2\pi i\zeta$, and integrate along the circular circumference γ_{00} in order to obtain relations that connect the parameter λ to the applied load p.

As a result we obtain [4]

$$
\alpha_0 + \sum_{k=1}^{\infty} \alpha_{4k}\lambda^{4k} r_{0,2k-1} = B + C + B\ln\lambda\left[1 + \sum_{k=1}^{\infty} A_{4k}\lambda^{4k} r_{0,2k-1}\right] \tag{17}
$$

The boundary conditions on the circumference of the hole L_{mn} (1) and the flow conditions determine A, B, and C.

Let us consider certain particular cases.

Tresk–St. Venant or Guber–Mises Plasticity Conditions. Suppose we have $|\sigma_\theta - \sigma_r| = 2K$ (K is the plasticity constant) in the plastic zone.

In this case we have, in accordance with Eqs. (1) and (2),

$$
A = 0, \quad B = \varepsilon K, \quad C = -\frac{p}{2} - \frac{\varepsilon K}{2}(1 + 2\ln R) \tag{18}
$$

Here, $\varepsilon = \pm 1$ is taken from physical concepts.

The results of a calculation in the first two approximations are given in Table 1.

Figure 1 depicts dependences (solid lines) of the parameter λ on the magnitude of the applied load p/K for certain values of the hole radius R = 0.5, 0.4, 0.3, 0.2, and 0.1 (curves 1-5).

Nonhomogeneously plastic material. Suppose that the plasticity condition now has the form [5]

$$
\sigma_\theta - \sigma_r = 2\left[K_0 + K_1\left(\frac{R}{r}\right)^2\right] \tag{19}
$$

Here, K_0 and K_1 are the material constants.

This plasticity condition can be considered as an ordinary Tresk – St. Venant condition with yield limit depending on the radius. In this case, we have, in accordance with Eqs. (1) and (2),

$$
A = -K_1 R^2, \quad B = K_0, \quad C = \frac{1}{2}(K_1 - K_0 - p - 2K_0\ln R) \tag{20}
$$

Results of a calculation to a second approximation are given in Table 2 for values of the nonhomogeneity parameters (K_1R^2/K_0) of 0.09 and -0.045.

The dependences of λ on the magnitude of the load p/K_0 for R = 0.3 are depicted in Fig. 1 by curves 6 and 7 for comparison for the values of K_1/K_0 (the nonhomogeneity parameters) of 1 and -0.5, respectively.

Exponential Yield Condition. Suppose the plasticity condition has the form

$$
\sigma_r - \sigma_\theta = 2K\left[1 - \exp\left(-\frac{\sigma_0}{K} + \frac{\sigma_r + \sigma_\theta}{2K}\right)\right] \tag{21}
$$

Here $K > 0$, and $\sigma_0 > 0$ are the material constants, which are of the same dimensions as the stresses.

This yield condition describes the limiting state of certain rocks [6]. In this case we have, in accordance with Eqs. (1) and (2),

$$
A = -Ke^{-2}t^{-2}R^2, \quad B = -K, \quad C = \frac{\sigma_0}{2} - K\ln tR^{-1} \tag{22}
$$

TABLE 1

λ	0.2	0.3	0.4	0.5	0.6	0.7
			First approximation			
β_2/K	1.0000	1.0000	1.0006	1.0026	1.0059	1.0431
β_6/K	0.0028	0.0142	0.0425	0.0878	0.1298	0.2671
A_4	—0.0028	—0.0142	—0.0424	—0.0876	—0.1289	—0.2634
α_4/K	0.0009	0.0047	0.0141	0.0291	0.0424	0.0956
			Second approximation			
β_2/K	1.0000	1.0000	1.0006	1.0026	1.0059	1.0634
β_6/K	0.0028	0.0142	0.0425	0.0879	0.1299	0.2423
β_{10}/K	0	0.0003	0.0022	0.0083	0.0084	0.0147
A_4	—0.0028	—0.0142	—0.0424	—0.0876	—0.1294	—0.2357
A_8	0	0	—0.0004	—0.0006	0.0085	0.0208
α_4/K	0.0009	0.0047	0.0141	0.0291	0.0425	0.1236
α_8/K	0	0	0.0002	0.0003	—0.0036	—0.0107

TABLE 2

λ	0.2	0.3	0.4	0.5	0.6	0.7
β_2/K_0	3.25	2.0004	1.5654	1.3664	1.2625	1.1957
β_6/K_0	0.0009	0.017	0.0515	0.1028	0.1472	0.2847
β_{10}/K_0	0	0.0007	0.0044	0.0134	0.0101	0.0191
A_4	—0.0092	—0.0284	—0.0663	—0.1192	—0.1622	—0.2901
A_8	0	—0.0001	—0.0006	—0.0008	—0.0109	0.0207
α_4/K_0	0.0031	0.0095	0.0221	0.0396	0.0532	0.1471
α_8/K_0	0	0	0.0002	0.0003	—0.0046	—0.0127
β_2/K_0	—0.125	0.5	0.7189	0.8213	0.8785	0.9147
β_6/K_0	—0.0005	0.0085	0.0339	0.0771	0.119	0.2213
β_{10}/K_0	0	0	0.0012	0.006	0.007	0.0132
A_4	0.0003	—0.0071	—0.0305	—0.0717	—0.113	—0.2149
A_8	0	0	—0.0003	—0.0005	0.0073	0.0167
α_4/K_0	—0.0001	0.0024	0.0102	0.0238	0.0371	0.1138
α_8/K_0	0	0	0.0001	0.0002	—0.0031	—0.0093

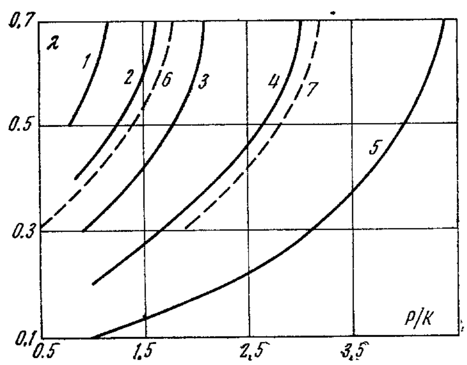

Fig. 1

where t is a constant and root of the equation

$$A = -Ke^{-2}t^{-2}R^2, \quad B = -K, \quad C = \frac{\sigma_0}{2} - K \ln tR^{-1}$$

More General Yield Conditions. Suppose the following yield conditions hold in the plastic zone [7]

$$K^{-1}(\sigma_0 + p) - 1 = e^{-2}t^{-2} + 2\ln t \quad (t > e^{-1}) \tag{23}$$

or

$$\sigma_r - \sigma_\theta = 2\left[K + \frac{b}{r^2} - K \exp\left(-\frac{\sigma_0}{K} + \frac{\sigma_r + \sigma_\theta}{2K}\right)\right] \tag{24}$$

In this case, the constants A, B, and C will have the following values, in accordance with Eqs. (1) and (2):

For condition (23)

$$\sigma_r - \sigma_\theta = 2\left[K + \frac{b}{r^2} - \frac{K}{r} \exp\left(-\frac{\sigma_0}{K} + \frac{\sigma_r + \sigma_\theta}{4K}\right)\right] \tag{25}$$

where t is a constant and root of the equation

$$A = b - Ke^{-2}t^{-2}R^2, \quad B = -K, \quad C = \frac{\sigma_0}{2} - K \ln tR^{-1}$$

For condition (24)

$$K^{-1}\left(\sigma_0 + p + \frac{b}{R^2}\right) - 1 = e^{-2}t^{-2} + 2\ln t \quad (t > e^{-1}) \tag{26}$$

where t is a constant and root of the equation

$$A = b - Ke^{-1}t^{-2}R^2, \; B = -K, \; C = \sigma_0 + K \ln (t^{-2}R^2)$$

Setting $\zeta = \lambda e^{i\theta}$ in Eq. (15) we obtain the elastoplastic equation of the boundary

$$K^{-1}\left(2\sigma_0 + p + \frac{b}{R^2}\right) - 1 + 2\ln R = 4\ln t + e^{-1}t^{-2} \quad (t > e^{-1})$$

$$r = |\omega(\lambda e^{i\theta})| = f(\theta)$$

To a first approximation,

$$r^2 = \lambda^2 (d - 2d_1 \cos 4\theta) \tag{27}$$

where

$$r_{\max} = \lambda\left[1 + A_4\left(-\frac{1}{3} + \lambda^4 \sum_{j=0}^{\infty} \frac{r_{2j,1}}{4j+1}\lambda^{4j}\right)\right] \tag{28}$$

$$r_{\min} = \lambda\left[1 + A_4\left(\frac{1}{3} + \lambda^4 \sum_{j=0}^{\infty} \frac{(-1)^j r_{2j,1}}{4j+1}\lambda^{4j}\right)\right] \tag{29}$$

The least load at which the hole circumference entirely encompases the plastic zone is determined from the condition $r_{\min} \geq R$. Equation (28) for $r_{\max} \leq 1$ allows the greatest load at which the plastic zones are tangent to each other, to be found.

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